Homogeneous Cone Fields: Affine-Invariant Orders (GiMLi 2018)

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Abstract

We discuss the theory of invariant cone fields as a framework for constructing novel partial orders on homogeneous spaces that are of interest in a variety of applications in information engineering and optimization. We illustrate the geometric ideas through an extended example involving new partial orders on the space of symmetric positive definite matrices.

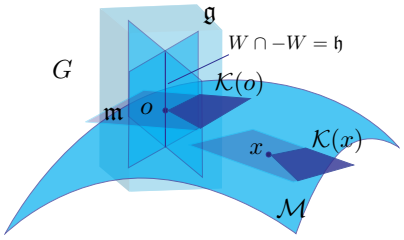

Figure 1. A homogeneous cone field $\mathcal{K}_{\mathcal{M}}$ on $\mathcal{M} = G/H$ arising as the projection of an invariant wedge field W_G on G generated by an Ad_H-invariant wedge $W \subset \mathfrak{g}$ that satisfies $W \cap -W = \mathfrak{h}$

1. Introduction

Well-defined notions of ordering of elements of a space are of fundamental importance to many areas of applied mathematics, including the theory of monotone functions and matrix means in which orders play a defining role. Partial orders play a key part in a wide variety of applications where one is interested in performing statistical analysis on sets of matrices, as well as in optimization theory through the notion of convexity. In such applications, the choice of order relation is often taken for granted. This choice, however, is of crucial significance and should be incorporated as a design question in problem-solving strategies whenever possible. Invariant cone fields provide a geometric approach to systematically construct 'natural' orders by connecting the geometry of the state space to the search for orders.

2. Affine-invariant partial orders

A smooth manifold $\mathcal{M}$ is said to be a *homogeneous space* if there exists a Lie group G that acts on $\mathcal{M}$ transitively. Such a space can be expressed as a quotient manifold $\mathcal{M} = G/H$ where H is the isotropy subgroup of G. Invariant cone fields on G/H arise as projections of suitable invariant wedge fields on G (Neeb, 1991; Hilgert et al., 1989). See Figure 1. Such a cone field $\mathcal{K}$ defines a *conal order* $\preceq_{\mathcal{K}}$, whereby $x \preceq_{\mathcal{K}} y$ if there exists a conal curve joining x to y.

The natural homogeneous geometry on the space of symmetric positive definite matrices S_n^+ is $GL(n)/O(n)$, induced by the transitive action of the general linear group. Invariant cone fields with respect to this structure have spectral characterizations and define *affine-invariant* partial orders (Mostajeran & Sepulchre, 2018a).

2.1. Invariant differential positivity

A map on a manifold is said to be *monotone* if it preserves an order structure. This important property is characterized locally by *invariant differential positivity*, i.e. differential positivity with respect to an invariant cone field (Mostajeran & Sepulchre, 2018b). Our framework provides a powerful, insightful, and testable approach for studying monotonicity on homogeneous spaces.

2.2. Geodesics as conal curves on S_n^+

The conal order induced by a cone field on a manifold is generally difficult to characterize. We discuss how invariance can be used to generate conal orders that are 'tractable'. In particular, we show that a pair of matrices are ordered with respect to an affine-invariant cone field on S_n^+ if and only if the geodesic between them is conal (Mostajeran & Sepulchre, 2018a).

3. Ongoing research

We may also present ongoing research and new results on the application of the theory to the homogeneous spaces of symmetric matrices of fixed inertia. Links with Finslerian geometry and related applications may also be explored.

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